On the Theory of Partitions

Applications of Dyson's "Rank" & "Crank"

Nicholas Wheeler
March 2019

Introduction. Freeman Dyson (1923–) was a scholar at Winchester College[1] when he first became aware of the remarkable accomplishments of Srinivasa Ramanujan (1889–1920). When he placed first in a mathematical competition, for which the prize was the book of his choice, Dyson selected Hardy & Wright's *An Introduction to Number Theory*, the first edition of which had been published the year before (1938). Hardy, of course, had been Ramanujan's principal mentor and collaborator.

In September of 1941, Dyson entered Trinity College, Cambridge as a seventeen-year-old undergraduate, where his teachers—in wartime decimated classes of seldom more than three students—were John Littlewood, Abram Besicovitch, P.A.M. Dirac. . . and Hardy himself. Soon after his arrival at Trinity, Dyson—first time away from home—began what became his persistent practice of addressing to his parents (and, after their demise, to his sister) very frequent, engaging and informative accounts of his activities. Those letters survive. An annotated selection, dating from October 1941 to April 1978, has been published as *Maker of Patterns: An Autobiography in Letters* (2018).

On Tuesday, November 10, 1942, Dyson—in the only one of his letters that contained even rudimentary mathematical material—described the problem to which he had devoted most of his time since the preceeding Thursday. The

[1] Dyson attended Winchester College (founded 1382) from 1936 until 1941. He speaks of his experience there always with warm nostalgia, and has claimed that he acquired there the interests, values and style that have informed his entire career. Julian Havil—author of a series of brilliant expository books, starting with *Gamma: Exploring Euler's Constant* (2003, with Foreword by Freeman Dyson)—for thirty years taught mathematics at Winchester.

problem marked his first engagement with what he later called his first love: the theory of partitions. Concerning the origin of the problem:

In 1919, Ramanujan—after study of a list of the number $p(n)$ of unrestricted partitions of n, from $p(1) = 1$ to $p(200) = 3972999029388$, that had recently been constructed by Major MacMahon—observed[2] that

$$\begin{array}{llllll}
(1) & p(4)\;, & p(9)\;, & p(14), & p(19), & \ldots \equiv 0 \pmod{5}\\
(2) & p(5)\;, & p(12), & p(19), & p(26), & \ldots \equiv 0 \pmod{7}\\
(3) & p(6)\;, & p(17), & p(28), & p(39), & \ldots \equiv 0 \pmod{11}\\
(4) & p(24), & p(49), & p(74), & p(99), & \ldots \equiv 0 \pmod{25}\\
\vdots & & & & & \\
(9) & p(116), & \ldots & & & \equiv 0 \pmod{121}\\
(10) & p(99)\;, & \ldots & & & \equiv 0 \pmod{125}
\end{array}$$

and on that evidence conjectured that if $A = 5^a 7^b 11^c$ and $24B \equiv 1 \pmod{A}$ then

$$p(An + B) \equiv 0 \pmod{A} \quad : \quad n = 0, 1, 2, \ldots \tag{1}$$

Most frequently encountered of those "Ramanujan congruences" are

$$p(5n + 4) \equiv 0 \pmod{5} \tag{2.1}$$

$$p(7n + 5) \equiv 0 \pmod{7} \tag{2.2}$$

$$p(11n + 6) \equiv 0 \pmod{11} \tag{2.3}$$

amd it was those that were of particular interest to Dyson. Ramanujan was able to construct proofs of the first two of those congruences, and to sketch proofs of $p(25n + 24) \equiv 0 \pmod{25}$ and of $p(49n + 47) \equiv 0 \pmod{49}$, but was not able to prove (1), which is, in fact, not valid as it stands: in the 1930s, S. Chowla, working from an expanded version of MacMahon's list, looked (in the case $n = 0$) to the conjectured congruence $p(7^3 n + 243) \equiv 0 \pmod{7^3}$ and noticed that, while indeed $24 \times 243 \equiv 0 \pmod{7^3}$,

$$p(243) = 133978259344888 \not\equiv 1 \pmod{7^3}$$

Correct modifications of (1) were devised by G. N. Watson in 1938 and by (Dyson's college friend) Oliver Atkin in 1967.

Dyson's invention and first application of "rank." Dyson defined the "rank" $r(\pi)$ of a partition π of n to be

$$r(\pi) = \text{largest element} - \text{number of elements}$$

For example, the partitions of 4 (in lexicographic order, as they are supplied by *Mathematica*) are $\{4\}, \{3, 1\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}$, and their respective ranks are $\{3, 1, 0, -1, -3\}$. Evidently, for arbitrary n one has

$$r_{\max} = +(n-1), \text{realized at } \{n\}$$

$$r_{\min} = -(n-1), \text{realized at } \{1, 1, 1, \ldots, 1\}$$

[2] My source here is George Andrews, *Theory of Partitions* (1976), pages 159–161.

Dyson noticed that

$$\{3,1,0,-1,-3\} \equiv \{3,1,0,4,2\} \pmod 5$$

and that $\{3,1,0,4,2\}$ is a permutation of the $p(4)=5$ numbers $\{0,1,2,3,4\}$. That observation acquires interest when—taking inspiration from (2.1)—one looks to the ranks (mod 5) of the $p(9)=30$ partitions of $5+4$. Those (which Dyson computed by hand, but which are instantly supplied by *Mathematica*) are found to comprise a (mysteriously ordered) 6-fold replication of $\{0,1,2,3,4\}$. Proceeding similarly, Dyson found that the ranks (mod 5) of the $p(14)=135$ partitions of $5\times 2+4=14$ comprise a 27-fold replication of $\{0,1,2,3,4\}$, and that the ranks (mod 5) of the $p(19)=490$ partitions of $5\times 3+4=19$ comprise a 98-fold replication of $\{0,1,2,3,4\}$.

Dyson conjectured, on the basis of this experimental evidence, that the partitions of $5n+4$, when subjected to the "ranks (mod 5)" process, always (*i.e.*, for all n) produce an N-fold replication of $\{0,1,2,3,4\}$. Ramanujan's first congruence $p(5n+4)\equiv 0 \pmod 5$ would then follow as an immediate consequence.

The conjecture gains plausibility/interest when one looks to the second congruence: $p(7n+5)\equiv 0 \pmod 7$. The partitions of 5 are $p(5)=7$ in number:

$$\{5\},\{4,1\},\{3,2\},\{3,1,1\},\{2,2,1\},\{2,1,1,1\},\{1,1,1,1,1\}$$

and when subjected to the "ranks (mod 7)" process produce $\{4,2,1,0,6,5,3\}$, which is a permutation of $\{0,1,2,3,4,5,6\}$. The partitions of $7+5=12$ are $p(12)=77$ in number, and when subject to that process produce a 10-fold replication of $\{0,1,2,3,4,5,6\}$. The partitions of $7\times 2+5=19$ are (as previously noted) 490 in number, and by that process produce a 70-fold replication.[3] The second Ramanujan congruence follows immediately from the conjecture that the pattern persists.

The satisfied optimism with which Dyson reported his accomplishment to his parents may owe something to the circumstance that he had yet to examine the third congruence $p(11n+6)\equiv 0 \pmod{11}$, which—though it shares the structure

$$p\big(p(B)n+B\big)\equiv 0 \big(\bmod p(B)\big)$$

of the first two congruences—Ramanujan himself had failed to prove. As it happens, and as Dyson was well aware then he published his discovery,[4] his method *fails* when applied to the third congruence, and for a very simple reason.

[3] All of which the 18-year-old Dyson worked out by hand, in what must have been a very busy five days.

[4] "Some guesses in the theory of partitions," Eureka (Cambridge) **8**, 10–15 (1944). That paper—Dyson's fifth publication—is reproduced in *Selected Papers of Freeman Dyson, with Commentary* (1996), pages 51–56, with richly informative commentary on pages 2–5. Eureka was a student publication.

The partitions of 6 are $p(6) = 11$ in number and, when subjected (most conveniently with the assistance of *Mathematica*) to the "ranks (mod 11)" process, produce $\{5, 3, 2, 1, 1, 0, 5, 5, 4, 3, 1\}$, in which there are repetitions, and which is therefore *not* a permutation of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

In "Some guesses..."[4] Dyson phrased his remarks in terms of the functions

$$\begin{aligned} N(m,n) &= \text{number of partitions of } n \text{ with rank } m \\ N(m,q,n) &= \text{number of partitions of } n \text{ with rank} \equiv m \pmod q \end{aligned}$$

He uses numerical evidence to develop patterns displayed by those functions, and—drawing heavily upon material borrowed from Hardy & Wright—casts those patterns in the language of generating functions. He is led from those exposed patterns to this statement: "I hold in fact that there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and [shall write]

$$M(m,q,n) = \text{number of partitions of } n \text{ with crank} \equiv m \pmod q\text{"}$$

He proceeds to list anticipated properties of the function $M(m,q,n)$, and concludes his paper with these words: "Whether these guesses are warrented by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe that the "crank" is unique among arithmetical function in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!."[5]

Discovery of "crank." Finally, in 1988, after an interval of 44 years, George Andrews,[6] working in collaboration with his student Frank Garvan, devised a definition of "crank" which possessed all of Dyson's anticipated properties, the

[5] Dyson's paper—which provides only conjectures, "guesses" (no proofs)—begins with these words: "Professor Littlewood, when he makes use of an algbraic identity, always saves himself the trouble of proving it; he maintains that an identity, if true, can be verified in a few lines by anybody obtuse enough to feel the need of verification. My objective in the following pages is to confute this assertion. ... I state certain properties of partitions which I am unable to prove: these guesses are then transformed into algebraic identities which are also unproved (though supported by numerical evidence); finally, I indulge in some even vaguer guesses which I am not only unable to prove but unable to state. I think this should be enough to disillusion anyone who takes Professor Littlewood's innocent view of the difficulties of algebra." Dyson's conjectures, so far as they relate to the first pair of Ramanujan congruences, were proven correct by his friends Oliver Atkin and Peter Swinnerton-Dyer in 1953.

[6] Andrews (1938–), leading expert on the theory of partitions[2] and discoverer of Ramanujan's "Lost Notebook," grew up in Salem, Oregon and graduated from Oregon State University.

structure of which Dyson reportedly found to be quite surprising: the Andrews/Garvan definition of the “crank” $c(\pi)$ of a partition π of n reads

$$c(\pi) = \begin{cases} \text{largest element} & : \quad \omega(\pi) = 0 \\ \mu(\pi) - \omega(\pi) & : \quad \omega(\pi) > 0 \end{cases}$$

where

$$\omega(\pi) = \text{number of 1's in } \pi$$
$$\mu(\pi) = \text{number of elements} > \text{than } \omega(\pi)$$

Evidently, for arbitrary n one has

$$c_{\max} = +n, \text{ realized at } \{n\}$$
$$c_{\min} = -n, \text{ realized at } \{1, 1, 1, \ldots, 1\}$$

On page 2 we looked to the partitions $\{4\}, \{3, 1\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}$ of 4 and found their respective ranks to be $\{3, 1, 0, -1, -3\}$. Their respective cranks are $\{4, 0, 2, -2, -4\}$. Where we had

$$\{3, 1, 0, -1, -3\} \equiv \{3, 1, 0, 4, 2\} \pmod 5$$

as a description of the process “ranks (mod 5)” we now have

$$\{4, 0, 2, -2, -4\} \equiv \{4, 0, 2, 3, 1\} \pmod 5$$

as a description of the process “cranks (mod 5),” a process that has again produced a permutation of $\{0, 1, 2, 3, 4\}$. Looking to the 7 partitions of 5, we were led by ranks (mod 7) to

$$\{4, 2, 1, 0, -1, -2, -4\} \equiv \{4, 2, 1, 0, 6, 5, 3\} \pmod 7$$

and are led by cranks (mod 7) to

$$\{5, 0, 3, -1, 1, -3, -5\} \equiv \{5, 0, 3, 6, 1, 4, 2\} \pmod 7$$

which again a permutation of $\{0, 1, 2, 3, 4, 5, 6\}$. Nothing thus far appears to be lost were we to abandon “rank” in favor of “crank.” But in connection with the third congruence $p(11n + 6) \equiv 0 \pmod{11}$ it was remarked already on page 4 that ranks (mod 11) supplies

$$\{5, 3, 2, 1, 1, 0, -1, -1, -2, -3, -5\} \equiv \{5, 3, 2, 1, 1, 0, 10, 10, 9, 8, 6\} \pmod{11}$$

which is *not* a permutation of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Cranks (mod 11), on the other hand, produces

$$\{6, 0, 4, -1, 3, 1, -3, 2, -2, -4, -6\} \equiv \{6, 0, 4, 10, 3, 1, 8, 2, 9, 7, 5\} \pmod{11}$$

where $\{6, 0, 4, 10, 3, 1, 8, 2, 9, 7, 5\}$ *is* such a permutation; the repetitions that afflicted ranks (mod 11) have been avoided. Application of cranks (mod 11) to the 297 partitions of $11 \times 1 + 6 = 17$ produces 27 replications of $\{0, 1, 2, \ldots, 10\}$; application to the 3718 partitions of $11 \times 2 + 6 = 28$ produces 338 replications.

We conclude that, on replacing "rank" by "crank," Dyson's argument serves not simply to prove but to *explain* all three of Ramanujan's congruences (2). Whether it can be used to establish all of Ramanujan's conjectured congruences (1) remains an open question, one not considered by Dyson.

Alternatives to the Andrews-Garvan definition of "crank" have been proposed. It has been reported that a close reading of the Lost Notebook shows Ramanujan himself to have been aware of partitions dissections similar to those produced by crank.

Rank revisited: Dyson's "New symmetries..." "Unrestricted partitions of n" are most commonly understood to be sets π of positive integers that sum to n. But contexts do arise—for exmple,

$$\begin{aligned}(x+y+z)^3 =& x^3y^0z^0 + x^0y^3z^0 + x^0y^0z^3\\ &+ 3x^2y^1x^0 + 3x^2y^0z^1 + 3x^0y^2x^1\\ &+ 3x^1y^2z^0 + 3x^1y^0z^2 + 3x^0y^1z^2 + 6x^1y^1z^1\end{aligned}$$

where (if order is disregarded) the partitions of 3 are $\{3,0,0\},\{2,1,0\},\{1,1,1\}$ —where it is natural to include one or more 0's among the elements of π. It becomes necessary, therefore, to distinguish *unrestricted positive partitions* from *unrestricted non-negative partitions*. Or, if we understand the "order" of a non-negative partition to be its number of 0 elements, to distinguish partitions of 0-order from those with order > 0.

Dyson, on sabbatical in 1968 from the Institute for Advanced Study, taught astrophysics at Yeshiva University. It had been his habit to "take occasional short holidays from physics and return to my first love, the theory of numbers," and it was after such a digression that (April 21, 1968) he wrote "Today I discovered a little theorem which gave me some intense moments of pleasure. It is beautiful and fell into my hand like a jewel from the sky." By "a little theorem" he refers to the substance of "A new symmetry of partitions,"[7]

The generative idea is explore the (surprisingly rich) body of material that results when "rank" is brought to the study of *non-negative* partitions. The discussion culminates in an elementary derivation of Euler's Pentagonal Number Theorem (1775).

We look, by way of preparation, to the ranks of *positive* partitions. Let

$$p(m,n) = \text{number of positive partitions of } n \text{ with rank } m$$

As previously remarked,

$$\begin{aligned}p(m<-n,n) = p(m>n,n) &= 0\\ p(m=1-n,n) = p(m=n-1,n) &= 1\end{aligned}$$

[7] J. Comb. Theory **7**, 56–61 (1969), reprinted in *Selected Papers*, 109–114. The paper feeds on numerical evidence, the production of which—in those pre-computer days—must have involved a good deal of fairly heavy labor, so "fell from the sky" can hardly be a fair description of Dyson's experience.

0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	1	1	1	1	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	1	1	2	1	2	1	1	0	1	0	0	0	0	0
0	0	0	0	1	0	1	1	2	1	3	1	2	1	1	0	1	0	0	0	0
0	0	0	1	0	1	1	2	2	3	2	3	2	2	1	1	0	1	0	0	0
0	0	1	0	1	1	2	2	3	3	4	3	3	2	2	1	1	0	1	0	0
0	1	0	1	1	2	2	4	3	5	4	5	3	4	2	2	1	1	0	1	0

TABLE 1: *Tabulated values of* $p(m,n)$. *Columns are labeled* $m=-10,-9,\ldots,,+9,+10$, *rows are labeled* $n=0,1,2,\ldots,10$.

Note the bilateral symmetry with respect to the $m=0$ column:

$$p(-m,n)=p(+m,n)$$

Every partition π of n possess *some* rank, so

$$\sum_m p(m,n)=p(n)$$

which checks out; for example, the numbers on the bottom row sum to $42=p(10)$.

Appending a 0 to a partition decreases its rank by 1. From the partitions $\{4\},\{3,1\},\{2,2\},\{2,1,1\},\{1,1,1,1\}$ we obtained the $n=4$ row in TABLE 1. Appending solitary 0's produces $\{4,0\},\{3,1,0\},\{2,2,0\},\{2,1,1,0\},\{1,1,1,1,0\}$ —distinct (non-negative) partitions of 4 (the partition of order 1), and sends

0	0	0	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0

to

0	0	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0

Every n possesses non-negative partitions of every order. We are led thus (in the case $n=4$, adjoining 0's one at a time) to construct

0	0	0	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Adding the numbers in the respective columns, we obtain a description

$$5\quad 5\quad 5\quad 5\quad 5\quad 5\quad 5\quad 5\quad 4\quad 4\quad \mathbf{3}\quad 2\quad 1\quad 1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0$$

of the number of members of the set of non-negative partitions (of unrestricted order) of $n = 4$ that have rank m.[8] Proceeding in this way (with the valuable assistance of *Mathematica*) we construct the following tabulation of values assumed by

$$q(m,n) = \text{number of non-negative partitions of } n \text{ with rank } m$$

1	1	1	1	1	1	1	1	**0**	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	1	1	0	0	0	0	0	0	0
3	3	3	3	3	3	3	2	2	1	1	0	0	0	0	0	0
5	5	5	5	5	5	4	4	3	2	1	1	0	0	0	0	0
7	7	7	7	7	6	6	5	4	3	2	1	1	0	0	0	0
11	11	11	11	10	10	9	8	6	5	3	2	1	1	0	0	0
15	15	15	14	14	13	12	10	9	6	5	3	2	1	1	0	0
22	22	21	21	20	19	17	15	12	10	7	5	3	2	1	1	0

TABLE 2: *Tabulated values of $q(m,n)$. Columns are labeled $m = -8, -7, \ldots, , +7, +8$, rows are labeled $n = 0, 1, 2, \ldots, 8$. Boldface* **0** *idenifies the $m = 0$ column.*

The design of the $n = 0$ row reflects the fact that, while 0 has no positive partition, it has any number of non-negative partitions, with $\text{rank}(\{0\}) = -1$, $\text{rank}(\{0,0\}) = -2$, etc.

Jewels from the sky. These are patterns latent in the infinite table of which TABLE 2 is a finite sample, patterns evident to the eye of a "maker of patterns" such as Dyson called himself in the title of his autobiography, properties of the function $q(m,n)$, "new symmetries of partitions."

$\boxed{\text{Natural emergence of } p(n)}$ The values assumed by $p(n)$ at $1, 2, 3, \ldots, 8$ are 1, 2, 3, 5, 7, 11, 15, 22. We see those numbers marching down the antidiagonal that terminates at $(m,n) = (1,0)$, and along the horizontal rows that terminate at that antidiagonal. In short,

$$q(m,n) = p(n) \quad : \quad m \leqslant -(n-1) \tag{3.1}$$

Also

$$q(m,n) = 0 \quad : \quad m \geqslant n \tag{3.2}$$

Less obviously (look to the region bounded on the left by by that antidiagonal and on the right by the diagonal that terminates at $(m,n) = (-1,0)$)

$$q(m,n) + q(1-m,n) = q(1+m,n) + q(-m,n) = p(n) \tag{3.3}$$

[8] The boldface **3** locates the position of $m = 0$.

from which it follows in particular that

$$q(1,n) + q(0,n) = p(n) \tag{3.4}$$

Equations (3) hold at/below the point $(m,n) = (0,1)$ where the diagonal and antidiagonal intersect; *i.e.*, for $n \geqslant 1$. That is because at $n = 0$ we encounter an anomaly: the table supplies $q(0,0) = q(1,0) = 0$ while, by analytically natural convention,[9] $p(0) = 1$.

$\boxed{\text{“Slant symmetry”}}$ At issue here is the symmetry illustrated by the parenthetic entries in the following repetition of TABLE 2:

(1)	1	1	1	1	1	1	1	**0**	0	0	0	0	0	0	0	0
1	1	(1)	1	1	1	1	1	1	0	0	0	0	0	0	0	0
2	2	2	2	(2)	2	2	1	1	0	0	0	0	0	0	0	0
3	3	3	3	3	3	(3)	2	2	1	1	0	0	0	0	0	0
5	5	5	5	5	5	4	4	(3)	2	1	1	0	0	0	0	0
7	7	7	7	7	6	6	5	4	3	(2)	1	1	0	0	0	0
11	11	11	11	10	10	9	8	6	5	3	2	(1)	1	0	0	0
15	15	15	14	14	13	12	10	9	6	5	3	2	1	(1)	0	0
22	22	21	21	20	19	17	15	12	10	7	5	3	2	1	1	(0)

Note that the symmetry is “slant bilateral” with respect (not to the $m = 0$ column, but) to the $m = -1$ column. Slant symmetry[10] can be formulated

$$q(m,n) = q(-m-2, n-m-1) \tag{4}$$

But by (3.3)

$$= p(n) - q(1-m,n)$$

so $q(1-m,n) = p(n) - q(-m-2, n-m-1)$, which by notational adjustment becomes

$$q(m,n) = p(n) - q(m-3, n+m-2) \tag{5.1}$$

Arguing similarly from $q(-m-2, n-m-1) = p(n) - q(1-m,n)$, we obtain

$$q(m,n) = p(n-m-1) - q(m+3, n-m-1) \tag{5.2}$$

Mathematica confirms the validity of (5.1) whenever $n + m - 2 \geqslant 0$, and that of (4) and (5.2) whenever $n - m - 1 \geqslant 0$.

[9] The situation here is analogous to the convention $0! = 1$, which is made natural by the strtucture of Taylor’s theorem and by the value assumed by the right side of $n! = \Gamma(n+1)$ at $n = 0$.

[10] The “adjoint” π' of a positive partition π of n is produced by transposing the Ferrer diagram of π. Non-positive partitions do not possess Ferrer diagrams. Dyson, however, has invented a work-around that permits him to say that “slant symmetry is simply adjoint symmetry.”

$\boxed{\text{Recursive constructions of } q(m,n)}$ Feeding (5.1) into itself once/twice gives

$$\begin{aligned} q(m,n) &= p(n) - p(n+m-2) + q(m+6, n+2m-7) \\ &= p(n) - p(n+m-2) + p(n+2m-7) - q(m+6, n+3m-15) \end{aligned}$$

whence

$$q(m,n) = \sum_{k=0} (-1)^k p(n+km-\phi(k)) \tag{6.11}$$

Assume $\phi(k) = \alpha k^2 + \beta k + \gamma$, require $\phi(1)=2$, $\phi(2)=7$, $\phi(3)=15$ and get

$$\phi(k) = \tfrac{1}{2}k(3k+1) \tag{6.12}$$

Proceeding similarly from (5.2) we find

$$\begin{aligned} q(m,n) = p(n-m-1) - p(n-2m-5) &+ p(n-3m-12) \\ &- q(m+9, n-3m-12) \end{aligned}$$

whence

$$q(m,n) = \sum_{k=1} (-1)^{k-1} p(n-km-\psi(k)) \tag{6.21}$$

where $\psi(k) = \alpha k^2 + \beta k + \gamma$ and $\psi(1)=1$, $\psi(2)=5$, $\psi(3)=12$ entail

$$\psi(k) = \tfrac{1}{2}k(3k-1) = \phi(-k) \tag{6.22}$$

One has $p(n<0)=0$, so the series (6.11) and (6.21) invariably terminate.

Typically (but not invariably) (6.1) and (6.2) assemble $q(m,n)$ in distinct ways. For example, both give

$$\begin{aligned} q(-1,10) &= p(10) - p(7) + p(1) \\ &= 42 - 15 + 1 = 28 \end{aligned}$$

But (6.1) produces

$$\begin{aligned} q(-5,10) &= p(10) - p(3) \\ &= 42 - 3 = 39 \end{aligned}$$

while (6.2) gives

$$\begin{aligned} q(-5,10) &= p(14) - p(15) + p(13) - p(8) + p(1) \\ &= 135 - 176 + 101 - 22 + 1 = 39 \end{aligned}$$

Their applications to slant-symmetric mates are, however, complementary. For example, in the case $q(-4,5) = q(2,8) = 7$ they produce (respectively)

$$q(-4,5) = \begin{cases} p(5) \\ p(8) - p(8) + p(5) \end{cases}$$
$$q(2,8) = \begin{cases} p(8) - p(8) + p(5) \\ p(5) \end{cases}$$

Figurate numbers. Interest in "figurate numbers," which arise when one looks to successively larger generations of Triangular/Square/Pentagonal/Hexagonal arrays of points, apparently extends back to the very beginnings of mathematics. The following display (due reportedly to Pythagorus)

$$\begin{array}{ll} T(1) = 1 & S(1) = 1 \\ T(2) = T(1) + 2 & S(2) = S(1) + 3 \\ T(3) = T(2) + 3 & S(3) = S(2) + 5 \\ T(4) = T(3) + 4 & S(4) = S(3) + 7 \\ T(5) = T(4) + 5 & S(5) = S(4) + 9 \end{array}$$

$$\begin{array}{ll} P(1) = 1 & H(1) = 1 \\ P(2) = P(1) + 4 & H(2) = H(1) + 5 \\ P(3) = P(3) + 7 & H(3) = H(2) + 9 \\ P(4) = P(3) + 10 & H(4) = H(3) + 13 \\ P(5) = P(4) + 13 & H(5) = H(4) + 17 \end{array}$$

illustrates the recursive construction of figurate numbers. In the Triangular case the additive terms (or "gnomon") increase by increments of 1, in the Square case by increments of 2, in the Pentagonal case by increments of 3, in the Hexagonal case by increments of 4, etc. Fitting the resulting low-order data to polynomials of lowest feasible order, we obtain

$$\begin{aligned} T(n) &= \tfrac{1}{2}n(n+1) \\ S(n) &= n^2 \\ P(n) &= \tfrac{1}{2}n(3n-1) = \tfrac{1}{3}T(3n-1) \\ H(n) &= n(2n-1) \end{aligned} \tag{7}$$

Which brings us to the point of this digression; namely, to the recognition that the numbers $\psi(k) = \frac{1}{2}k(3k-1)$ encountered at (6.22) are pentagonal numbers, while the numbers $\phi(k) = \frac{1}{2}k(3k+1) = \psi(-k)$ encountered at (6.12) are "generalized" pentagonal numbers. Here are the pentagonal numbers for $-5 \leqslant n \leqslant 5$:

$$\ldots, 40, 26, 15, 7, 2, 0, 1, 5, 12, 22, 35, \ldots$$

The relationship (7) between pentagonal and tringular numbers is holds even for negative values of n.

Generators of q(m,n). The discussion proceeds from (6) and from this elementary remark: if we write $H(x) = \sum_{n=0}^{\infty} p(n)x^n$ to generate the partition numbers $p(n)$, and if ν is a non-negative integer, then

$$\begin{aligned} H(x)x^\nu = \sum_{n=0}^{\infty} p(n)x^{n+\nu} &= \sum_{n=\nu}^{\infty} p(n-\nu)x^n \\ &= \sum_{n=0}^{\infty} p(n-\nu)x^n \quad \text{by } p(n) = 0 \ : \ n \leqslant 0 \end{aligned}$$

From (6.1) we are led therefore to construct

$$F(x,m) = H(x)\sum_{k=0}^{\infty}(-1)^k x^{-mk+\frac{1}{2}k(3k+1)} \tag{8.1}$$

and from (6.2) to construct

$$G(x,m) = H(x)\sum_{k=1}^{\infty}(-1)^{k-1} x^{mk+\frac{1}{2}k(3k-1)} \tag{8.2}$$

which we expect to serve as slant-symmetry-equivalent descriptions of the generator $\sum_{n=0}^{\infty} q(m,n)x^n$ of the numbers $q(m,n)$ that stand in the m^{th} column of TABLE 2.

Using *Mathematica* to check the accuracy of those expectations,[11] we find that

$$F(x,m) = \text{correct series} \sum q(m,n)x^n \text{ for all } m<0$$

At $m=0$ we encounter $q(0,0)=1$ instead of the correct $q(0,0)=0$:

$$F(x,0) = 1 + \text{correct series} \textstyle\sum_{n=1} q(0,n)x^n$$

At $m=1$ we encounter $q(1,0)=1$ instead of the correct $q(1,0)=0$:

$$F(x,1) = 1 + \text{correct series} \textstyle\sum_{n=1} q(1,n)x^n$$

and for $m>1$ we encounter periodically intrusive singular terms:

$$\begin{aligned}
F(x,2) &= && \text{correct series} \textstyle\sum_{n=0} q(2,n)x^n\\
F(x,3) &= -1/x && +\,\text{correct series} \textstyle\sum_{n=0} q(3,n)x^n\\
F(x,4) &= -1/x^2 && +\,\text{correct series} \textstyle\sum_{n=0} q(4,n)x^n\\
F(x,5) &= && \text{correct series} \textstyle\sum_{n=0} q(5,n)x^n\\
F(x,6) &= +1/x^5 && +\,\text{correct series} \textstyle\sum_{n=0} q(6,n)x^n\\
F(x,7) &= +1/x^7 && +\,\text{correct series} \textstyle\sum_{n=0} q(7,n)x^n\\
F(x,8) &= && \text{correct series} \textstyle\sum_{n=0} q(8,n)x^n\\
F(x,9) &= -1/x^{12} && +\,\text{correct series} \textstyle\sum_{n=0} q(6,n)x^n\\
F(x,10) &= -1/x^{15} && +\,\text{correct series} \textstyle\sum_{n=0} q(10,n)x^n
\end{aligned}$$

---

[11] In numerical work we much truncate the sums that enter into the definitions (8); we work actually from

$$\begin{aligned}
F(x,m;\alpha,\beta) &= \sum_{j=0}^{\alpha} p(j)x^j \sum_{k=0}^{\beta}(-1)^k x^{-mk+\frac{1}{2}k(3k+1)}\\
G(x,m;\alpha,\beta) &= \sum_{j=0}^{\alpha} p(j)x^j \sum_{k=1}^{\beta}(-1)^{k-1} x^{mk+\frac{1}{2}k(3k-1)}
\end{aligned}$$

with α,β set high enough to achieve stability/accuracy up through $n=10$.

Here I display the essential features of an extended list of singular terms:

$$\left(\begin{array}{lcccccccccccccccc}
m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16\\
\text{sign} & & & - & - & & + & + & & - & - & & + & + & & - & -\\
\text{exp} & & & 1 & 2 & & 5 & 7 & & 12 & 15 & & 22 & 26 & & 35 & 40
\end{array}\right)$$

Fitting the first three data points to a quadratic in m, we obtain

$$\text{exponent}(m) = \tfrac{1}{6}m(m-1) \equiv f(m)$$

which reproduces all the exponential data (and at $m = 2, 5, 8, 11, 14$ produces fractions). Singular terms are seen to arise if and only if either $m \equiv 0 \,(\text{mod}\, 3)$ (*i.e.*, when $m = 3p$, with $p = 1, 2 \ldots$) or $m \equiv 1 \,(\text{mod}\, 3)$ (*i.e.*, when $m = 3p + 1$), and to be absent when $m \equiv 2 \,(\text{mod}\, 3)$. It follows that the F-generator of $q(m, n)$ is (for $m > 1$) properly described by this modification of (8.1):

$$\begin{aligned}
&\mathcal{F}(x,m)\\
&\quad = H(x)\sum_{k=0}^{\infty}(-1)^k x^{-mk+\frac{1}{2}k(3k+1)}\begin{cases}\text{if } m<0 \text{ or } m\equiv 2\,(\text{mod}\,3)\\[6pt] -(-1)^p x^{-\frac{1}{6}m(m-1)} \text{ otherwise,}\\ \textit{i.e.}, \text{ if } m=3p \text{ or } m=3p+1\end{cases}
\end{aligned}\tag{9.1}$$

The situation with regard to $G(x, m)$ is complementary. We find

$$G(x,m) = \text{correct series} \sum q(m,n)x^n \text{ for all } m \geqslant 0$$

For negative values of m we periodically encounter spurious singular terms that enter with the signs and exponents described below:[12]

$$\left(\begin{array}{lcccccccccccccccc}
-m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16\\
\text{sign} & & + & + & & - & - & & + & + & & - & - & & + & + & \\
\text{exp} & & 1 & 2 & & 5 & 7 & & 12 & 15 & & 22 & 26 & & 35 & 40 &
\end{array}\right)$$

Noting that $\{-2, -5, -8, -11, \ldots\} \equiv 2 \,(\text{mod}\, 3)$, so are of the form $m = -3p+1$; that $\{-3, -6, -9, -12, \ldots\} \equiv 0 \,(\text{mod}\, 3)$, so are of the form $m = -3p$; and that $\{-1, -4, -7, , 10, \ldots\} \equiv 2 \,(\text{mod}\, 3)$, we conclude that the G-generator of $q(m, n)$ is properly described by this modification of (8.2):

$$\begin{aligned}
&\mathcal{G}(x,m)\\
&\quad = H(x)\sum_{k=1}^{\infty}(-1)^k x^{-mk+\frac{1}{2}k(3k+1)}\begin{cases}\text{if } m\geqslant 0 \text{ or } m\equiv 2\,(\text{mod}\,3)\\[6pt] +(-1)^p x^{-\frac{1}{6}m(m-1)} \text{ otherwise,}\\ \textit{i.e.}, \text{ if } m=-3p \text{ or } m=-3p+1\end{cases}
\end{aligned}\tag{9.2}$$

---

[12] Note that the entries in the third row of the following table are, relative to the preceeding table, shifted one unit to the left. That is because

$$f(m) = f(-(m-1))$$

Brightest of Dyson's "jewels from the sky." At $m = 0$ we have

$$F(x,0) = H(x)\sum_{k=0}^{\infty}(-1)^k x^{\frac{1}{2}k(k(3k+1))} = 1 + \sum_{n=1}^{\infty} q(0,n)x^n \tag{10.1}$$

where the $1(\neq q(0,0){=}0)$ was previously dismissed as an anomaly. On the other hand, we (by slant symmetry) have

$$\begin{aligned} G(x,0) &= H(x)\sum_{k=1}^{\infty}(-1)^{k-1} x^{\frac{1}{2}k(k(3k-1))} \\ &= -H(x)\sum_{-\infty}^{-1}(-1)^k x^{\frac{1}{2}k(3k+1)} = 0 + \sum_{n=1}^{\infty} q(0,n)x^n \end{aligned} \tag{10.2}$$

Subtracting (10.2) from (10.1), we have

$$H(x)\sum_{-\infty}^{\infty}(-1)^k x^{\frac{1}{2}k(3k\pm 1)} = 1$$

or

$$\frac{1}{H(x)} = \sum_{-\infty}^{\infty}(-1)^k x^{\frac{1}{2}k(3k\pm 1)} \tag{11}$$

where the sign choice in the exponent is arbitrary; since the sum is of the form $\sum_{-N}^{+N}$, reversing the sign simply reverses the sequence in which the terms are deployed.

Leibniz, already in 1674, had brought to the attention of J. Bernoulli the problem of counting the number of ways in which a positive integer can be written as a sum of such integers, but it was two questions that Phillippe Naudé le Jeune (French mathematician, 1684–1745) addressed to Euler in about 1740 that sparked what became the theory of partitions. Naudé's questions were relatively specific: "In how many ways can the number 50 be written as the sum of seven distinct positive integers?" and "In how many ways can the number 50 be written as the sum of seven positive integers, equal of unequal?" Euler approach those and related questions by the method of generating functions. One of the first fruits of that effort was[13]

$$H(x) \equiv \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \sum_{n=1}^{\infty} p(n)x^n \tag{12}$$

In the paper (1741) in which he reported that result, Euler also reported the experimental result

$$\begin{aligned} \frac{1}{H(x)} &= (1-x)(1-x^2)(1-x^3)\cdots \\ &= 1 - x^2 - x^5 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \\ &\qquad - x^{35} - x^{40} + x^{51} + x^{57} - \cdots \end{aligned} \tag{13}$$

of which he was able to supply a proof only many years later. The signs and

[13] It is this and similar results that motivate the convention $p(0) = 1$.

in (13) can be obtained by zig-zagging down the following table:

k	$(-1)^k \frac{1}{2}k(3k-1)$	$(-1)^k \frac{1}{2}k(3k+1)$
0	0	0
1	$-\ 1$	$-\ 2$
2	$+\ 5$	$+\ 7$
3	-12	-15
4	$+22$	$+26$
5	-35	-40
6	$+51$	$+57$
7	-70	-77
8	$+92$	$+100$

At (13) we have a statement of *Euler's pentagonal theorem*, proofs of which are usually fairly intricate.[14] At (11) Dyson has arrived by relatively elementary means at a statement of Euler's theorem, but his argument does not rise to the status of "proof"—though he speaks of it in those terms[15]—since it is based upon unproven extrapolations from patterns evident in a short table (Table 2) of computed data. Similarly, Dyson's highly suggestive account of where the first pair of Ramanujan congruences come from was based upon illustrative low-order calculations/conjectures related to those displayed in Table 1, and acquired the status of proof only with the work of Oliver Atkin and Peter Swinnerton-Dyer, a decade later.[5] It was Dyson's "rank" that generated both developments: here the ranks of non-negative integers, there the ranks of positive integers.

PART TWO

Crank analogs of Dyson's "new symmetries." Dyson wrote "New Symmetries. . . " in 1968, twenty years before Andrews and Garvan hit upon a definition of "crank" that met all the conditions imagined by Dyson in 1944. The invention of crank permitted construction of a variant of Dyson's elegantly simple "Some guesses. . . " argument that serves to explain not only the first pair but also the third of Ramanujan's congruences (2), as was discussed on pages 4–6. It becomes in this light natural to ask: What would have been the form assumed by "New Symmetries. . . " if Dyson had had in hand the Andrews-Garvan definition of crank? That is the issue to which I now turn.

---

[14] The theorem can be obtained as a corollary of an identity due to Jacobi which belongs properly to the theory of elliptic functions. Fabian Franklin (1853–1939, a Johns Hopkins student of Sylvester) devised a widely-admired elementary combinatorial proof in 1881. See §§19.9 & 19.11 in Hardy & Wright 6^{th} edition, 2008.

[15] "New symmetries. . . " concludes with these words: "This combinatorial derivation of Euler's formula is less direct, but perhaps more illuminating, than the well-known combinatorial proof by Franklin."

We encounter at the outset a small problem: if π is a positive partition of n, with rank m, then the ranks of $\{\pi,0\},\{\pi,0,0\},\ldots$ are $m-1,m-2,\ldots$ This is what gave rise to the "shift left" principle illustrated on page 7, that enabled us to extract from TABLE 1 the data recorded in TABLE 2, from the latter of which Dyson abstracted his "new symmetries." But from the Andrews-Garvan definition

$$c(\pi)=\begin{cases}\text{largest element} & : \quad \omega(\pi)=0\\ \mu(\pi)-\omega(\pi) & : \quad \omega(\pi)>0\end{cases}$$

$$\omega(\pi)=\text{number of 1's in }\pi$$

$$\mu(\pi)=\text{number of elements}>\text{than }\omega(\pi)$$

it follows that if $c(\pi)=m$ then so do $c(\{\pi,0\})=c(\{\pi,0,0\})=\cdots=m$. To restore the "left shift" principle, upon which all else hinges, we adopt this modified definition

$$c(\pi)=\begin{cases}\text{largest element} & : \quad \omega(\pi)=0\\ \mu(\pi)-\omega(\pi)-\zeta(\pi) & : \quad \omega(\pi)>0\end{cases}$$

$$\omega(\pi)=\text{number of 1's in }\pi$$

$$\mu(\pi)=\text{number of elements}>\text{than }\omega(\pi)$$

$$\zeta(\pi)=\text{number of 0's in }\pi$$

which reduces to the Andrews-Garvan crank when π is a positive partition of n; *i.e.*, when $\zeta(\pi)=0$.

With the assistance of *Mathematica* we construct this crank analog of TABLE 1:

0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	0	0
0	0	0	0	1	0	1	1	1	1	1	1	1	1	1	0	1	0	0	0	0
0	0	0	1	0	1	1	1	1	2	1	2	1	1	1	1	0	1	0	0	0
0	0	1	0	1	1	2	1	2	2	2	2	2	1	2	1	1	0	1	0	0
0	1	0	1	1	2	1	3	2	3	2	3	2	3	1	2	1	1	0	1	0
1	0	1	1	2	2	3	2	4	3	4	3	4	2	3	2	2	1	1	0	1

TABLE 3: *Tabulated values of the crank function* $p(m,n)$. *Columns are again labeled* $m=-10,-9,\ldots,,+9,+10$*, rows are labeled* $n=0,1,2,\ldots,10$.

To facilitate comparison of crank results with their rank counterparts I retain the symbols $p(m,n)$ and $q(m,n)$, but assign to them fresh crank-based meanings. Here, for example,

$$p(m,n)=\text{number of positive partitions of }n\text{ with }\textit{crank }m$$

Note that (except at $n = 1$) we again have bilateral symmetry with respect to the $m = 0$ column:

$$p(-m, n) = p(+m, n) \quad : \quad n \neq 1$$

And that again (because every positive partition π of n possess *some* crank)

$$\sum_m p(m, n) = p(n)$$

The "shift left and add" construction that led from Table 1 to Table 2 leads now from Table 3 to the following tabulation of values assumed by

$$q(m, n) = \text{number of non-negative partitions of } n \text{ with crank } m$$

0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	1	1	1	1	0	0	0	0	0	0
3	3	3	3	3	3	2	2	2	1	1	1	0	0	0	0	0
5	5	5	5	5	4	4	3	3	2	2	1	1	0	0	0	0
7	7	7	7	6	6	5	5	4	3	2	2	1	1	0	0	0
11	11	11	10	10	9	8	7	6	5	4	3	2	1	1	0	0
15	15	14	14	13	12	11	10	8	7	5	4	3	2	1	1	0
22	21	21	20	19	17	16	14	12	10	8	6	5	3	2	1	1

Table 4: *Tabulated values of* $q(m, n)$. *Columns are labeled* $m = -8, -7, \ldots, , +7, +8$, *rows are labeled* $n = 0, 1, 2, \ldots, 8$. *Boldface* **0** *idenifies the* $m = 0$ *column.*

We observe that (compare (3.1) and (3.2))

$$q(m, n) = p(n) \quad : \quad m \leqslant -n \tag{14.1}$$

$$q(m, n) = 0 \quad : \quad m \geqslant n \tag{14.2}$$

Less obviously $q(m, n) - q(1 + m, n) = p(m, n)$ and (compare (3.3))

$$q(m, n) + q(1 - m, n) = p(n) \tag{14.3}$$

In particular,

$$q(1, n) + q(0, n) = p(n) \tag{14.4}$$

Equations (14) hold for $n > 1$. These equations are formally identical to their rank-based counterparts, but here the definition of $q(m, n)$ is crank-based, so they describe *new* "new symmetries of partitions." Again we have "slant symmetry," as illustrated below, but—compare the table on page 6—it is "slant bilateral" with respect (not to the $m = -1$ column but) to the $m = 0$ column.

0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0
1	(1)	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
2	2	2	(2)	2	2	2	1	1	1	1	0	0	0	0	0	0
3	3	3	3	3	(3)	2	2	2	1	1	1	0	0	0	0	0
5	5	5	5	5	4	4	(3)	3	2	2	1	1	0	0	0	0
7	7	7	7	6	6	5	5	4	(3)	2	2	1	1	0	0	0
11	11	11	10	10	9	8	7	6	5	4	(3)	2	1	1	0	0
15	15	14	14	13	12	11	10	8	7	5	4	3	(2)	1	1	0
22	21	21	20	19	17	16	14	12	10	8	6	5	3	2	(1)	1

Slant symmetry can now be formulated

$$q(m,n) = q(-m, n-m) \tag{15}$$

But by (14.3) $\qquad = p(n) - q(1-m, n)$

so $q(1-m,n) = p(n) - q(-m, n-m)$, which by notational adjustment becomes

$$q(m,n) = p(n) - q(m-1, n+m-1) \tag{16.1}$$

Similarly, $q(-m, n-m) = p(n) - q(1-m, n)$ gives

$$q(m,n) = p(n-m) - q(m+1, n-m) \tag{16.2}$$

Equations (16) are the crank analogs of (5), but simpler.

From (16.1) if follows by recursion that

$$\begin{aligned} q(m,n) &= p(n) - p(n+m-1) + q(m-2, n+2m-3) \\ &= p(n) - p(n+m-1) + p(n+2m-3) - q(m-3, n+3m-6) \end{aligned}$$

whence

$$q(m,n) = \sum_{k=0} (-1)^k p(n + km - \phi(k)) \tag{17.11}$$

Assume $\phi(k) = \alpha k^2 + \beta k + \gamma$, require $\phi(1) = 1$, $\phi(2) = 3$, $\phi(3) = 6$ and get

$$\phi(k) = \tfrac{1}{2} k(k+1) \tag{17.12}$$

Proceeding similarly from (16.2) we find

$$q(m,n) = p(n-m) - p(n-2m-1) + p(n-3m-3) - p(n-4m-6) + \cdots$$

whence

$$q(m,n) = \sum_{k=1} (-1)^k p(n + km - \psi(k)) \tag{17.21}$$

$$\psi(k) = \tfrac{1}{2} k(k-1) = \phi(-k) \tag{17.22}$$

Note that the pentagonal numbers encountered at (6.12) have at (17.12) been replaced by triangular numbers.

Generators of the crank numbers q(m,n). We are led from (17.1) and (17.2), arguing as before, to introduce the crank functions[16]

$$F(x,m) = H(x)\sum_{k=0}^{\infty}(-1)^k\, x^{-km+\frac{1}{2}k(k+1)} \tag{18.1}$$

$$G(x,m) = H(x)\sum_{k=1}^{\infty}(-1)^{k-1}\, x^{km+\frac{1}{2}k(k-1)} \tag{18.2}$$

which are expected to provide alternative descriptions of $\sum_n q(m,n)x^n$, the generator of the numbers $q(m,n)$ that lie in the m^{th} column of TABLE 4. With *Mathematica*'s assistance we find that both of those generators produce the anticipated results *except* that both yield $q(m \leqslant 1, 0) = 1$, whereas in fact $q(m \leqslant 1, 0) = 0$. We find, moreover, that neither produces singular terms such as were encountered on page 12.

Dyson's derivation of Euler's pentagonal theorem was seen at (10) to gain essential leverage from an anomalous property of the rank-based generator $F(x,0)$:

expansion of $F(x,0)$ gives $q(0,0) = 1$ (incorrect), while
expansion of $G(x,0)$ gives $q(0,0) = 0$ (correct)

But the crank-based generators $F(x,0)$ and $G(x,0)$ are *both* anomalous

expansion of $F(x,0)$ gives $q(0,0) = 1$ (incorrect)
expansion of $G(x,0)$ gives $q(0,0) = 1$ (incorrect)

Deprived of that leverage, the crank-based theory yields no triangular analog of the pentagonal theorem.[17]

---

[16] For numerical work, use

$$F(x,m;\alpha,\beta) = \sum_{j=0}^{\alpha} p(j)x^j \sum_{k=0}^{\beta}(-1)^k\, x^{-km+\frac{1}{2}k(k+1)}$$

$$G(x,m;\alpha,\beta) = \sum_{j=0}^{\alpha} p(j)x^j \sum_{k=1}^{\beta}(-1)^{k-1}\, x^{km+\frac{1}{2}k(k-1)}$$

[17] The pentagonal theorem

$$(1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$$

arises as a corrollary of a Jacobi identity.[14] Another corrollary reads

$$\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = 1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + \ldots$$

which is as close as one can come to a "triangular theorem."

The preceeding material is conjectural, since based upon (convincingly patterned) extrapolations from the numertical data displayed in TABLES 3&4. "Anybody obtuse enough to feel the need of verification"[5] has his work cut out for him, for the proofs, on evidence of Atkin & Swinnerton-Dyer, promise to be difficult.

PART THREE

Implications of Dyson's "alternative crank." Twenty years after the publication (1969) of "New symmetries...," Dyson—drawing inspiration from some (Dyson-inspired) recent work (1986–88) by Andrews and (especially by) Garvan, whose assitance he acknowledges—published a third contribution[18] to this subject, one in which he actually purports to prove some things.

In that paper, Dyson adopts not the Andrews-Garvan definition of crank but (without attribution or a word of motivation) a novel construction of what I will (to emphasize the distinction) call "krank," which is defined by a set of three conditionals: given a partition $\pi = \{\pi_1, \pi_2, \ldots, \pi_s\} : \pi_{j+1} \geqslant \pi_j$ of an integer n, let

$$\begin{aligned}
t(\pi) &= \begin{cases} \pi_1 - \pi_2 & : \quad s > 1 \\ \pi_1 & : \quad s = 1 \end{cases} \\
d(\pi) &= \begin{cases} t - \pi_{t+1} & : \quad s > t \\ t & : \quad \text{otherwise} \end{cases} \\
\text{krank } k(\pi) &= \begin{cases} -s & : \quad t = 0 \\ d & : \quad t > 0 \end{cases}
\end{aligned}$$

We verify that krank possesses the properties that, in "Some guesses..., Dyson—44 years previously—had been led to require of it. For the 5 partitions of 4 we (with the assistaance of *Mathematica*) we obtain the kranks

$$\{4, 2, -2, 0, -1\} = \{4, 2, 3, 0, 1\}(\text{mod } 5)$$

which is a permutation of $\{0, 1, 2, 3, 4\}$. From the 30 partitions of $5 \times 1 + 4 = 9$ we obtain six copies of $\{0, 1, 2, 3, 4\}$, and from the 135 partions of $5 \times 2 + 4 = 14$ 27 copies, etc. For the 7 partitions of 5 we obtain

$$\{5, 3 - 1, 1, -3, 0, -5\} = \{5, 3, 6, 1, 4, 0, 2\}(\text{mod } 7)$$

which is a permutation of $\{0, 1, 2, \ldots, 6\}$. From the 77 partitions of $7 \times 1 + 5 = 12$ we obtain 11 copies, and from the 490 partitions of $7 \times 2 + 5 = 17$ we obtain 70 copies. Finally (and which in 1942 Dyson would have found more interesting), from the 11 partitions of 6 we obtain the kranks

$$\{6, 4, 2, 3, -2, -1, 1, -3, -4, 0, -6\} = \{6, 4, 2, 3, 9, 10, 1, 8, 7, 0, 5\}(\text{mod } 11)$$

which is a permutation of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. From the 297 partitions

[18] "Mappings and symmetries of partitions," J. Combinatorial Theory **A51**, 169–180 (1989), reprinted in *Selected Papers*, pages 115–126.

of $11{\times}1{+}6=17$ we obtain 27 copies, and from the 3718 partitions of $11{\times}2{+}6=28$ we obtain 338 copies. So much for the three Ramanujan congruences.

As those examples illustrate,

$$\begin{aligned} k_{\text{max}} &= +n, \text{ realized at } \{n\} \\ k_{\text{min}} &= -n, \text{ realized at } \{1,1,1,\ldots,1\} \end{aligned}$$

Writing

$$p(m,n) = \text{number of positive partitions of } n \text{ with krank } m$$

(where we have again assigned a new meaning to an old symbol) we consruct

0	0	0	0	0	0	0	0	0	0	**0**	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	0	0
0	0	0	0	1	0	1	1	1	1	1	1	1	1	1	0	1	0	0	0	0
0	0	0	1	0	1	1	1	1	2	1	2	1	1	1	1	0	1	0	0	0
0	0	1	0	1	1	2	1	2	2	2	2	2	1	2	1	1	0	1	0	0
0	1	0	1	1	2	1	3	2	3	2	3	2	3	1	2	1	1	0	1	0
1	0	1	1	2	2	3	2	4	3	4	3	4	2	3	2	2	1	1	0	1

TABLE 5: *Tabulated values of the krank function $p(m,n)$. Columns are again labeled $m=-10,-9,\ldots,,+9,+10$, rows are labeled $n=0,1,2,\ldots,10$. The 0^{th} row is a formal artifact/ placeholder, since krank of the empty set is undefined.*

of which only the 1^{st} row differs from TABLE 3: there $p(-1,1)=1, p(1,1)=0$; here the situation is reversed. Note the symmetry

$$p(-m,n) = p(m,n) \quad : \quad n > 1$$

Turning now from positive partitions $\pi = \{\pi_1,\pi_2,\ldots,\pi_s\}$ to partitions of the non-negative form $\{\pi,0,0,\ldots,0\}$, we observe that

$$k(\{0,0,\ldots,0\}) = -\text{ number of 0's}$$

and—to bring the essential "shift left" principle (page 7) into play—adopt this modified definition of krank:

$$k(\{\pi,0,0,\ldots,0\}) = k(\pi) - \text{number of 0's}$$

We are led then from TABLE 5 to the following tabulation of the values of (new definition)

$$q(m,n) = \text{number of non-negative partitions of } n \text{ with krank } m$$

1	1	1	1	1	1	1	1	**0**	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
2	2	2	2	2	2	2	1	1	1	1	0	0	0	0	0	0
3	3	3	3	3	3	2	2	2	1	1	1	0	0	0	0	0
5	5	5	5	5	4	4	3	3	2	2	1	1	0	0	0	0
7	7	7	7	6	6	5	5	4	3	2	2	1	1	0	0	0
11	11	11	10	10	9	8	7	6	5	4	3	2	1	1	0	0
15	15	14	14	13	12	11	10	8	7	5	4	3	2	1	1	0
22	21	21	20	19	17	16	14	12	10	8	6	5	3	2	1	1

TABLE 6: *Tabulated values of the krank function* $q(m,n)$. *Columns are labeled* $m = -8, -7, \ldots,, +7, +8$*, rows are labeled* $n = 0, 1, 2, \ldots, 8$. *Boldface* **0** *idenifies the* $m = 0$ *column.*

Here the $n = 0$ row differs from that of TABLE 4, and the $n = 1$ row differs at $m = 0$ and $m = 1$, but for $n > 1$ TABLES 4&6 are identical, and display therefore the same symmetries, give rise to the same generating functions (which are, however, now released from the exceptions described on page 19). These krank-based symmetries are distinct from the rank-based symmetries described in "New symmetries...," and fail as before to producer Euler's pentagonal theorem. But they are no longer conjectured extrapolations from a limited supply of computed data: for them Dyson is able in "Mappings..." to supply explicit proofs.